Electromagnetic Units and Equations

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Invited Paper

***Abstract*—Different systems of units are often used in electrical engineering and in physics, causing difficulties when the need arises to convert from one system of units to another. This can be a problem in certain areas of electromagnetism, as in microwave engineering when dealing with magnetic materials and devices, or with electromagnetic-wave propagation. It arises, for example, when converting between units or equations of the (rationalized, unsymmetrical) meter, kilogram, second, and ampere system and the (nonrationalized symmetrical) Gaussian system often used in physics. This paper solves the problem by means of *Gedanken* ("in-the-mind") experiments. You are invited to join "Alice in Wonderland," where anything is possible and every experiment has a happy outcome.**

***Index Terms*—cgs, dimensions, emu, esu, Gaussian, Maxwell's equations, MKSA, units.**

I. Introduction

YOU HAVE surely heard how Alice fell down the rabbit hole, at the bottom of which she landed in Wonderland. In Wonderland, things are possible that you could not even begin to imagine in the real world. *Gedanken* ("in-the-mind") experiments can be carried out in Wonderland that test and help to explain strange conundrums that crop up in our world. We shall see how some common misconceptions about electromagnetic units were thus resolved by the resourceful denizens of Wonderland. Alice is anxious that scientists and engineers everywhere may profit from these Wonderland experiments.

After her first adventure in Wonderland, Alice loved to go back and spend each vacation with her new-found friends there. The record of three of those vacations follows.

A. Vacation I

(From which Alice learns how to convert between rationalized and nonrationalized units and equations.)

Alice's first vacation in Wonderland was coming to an end. The Mad Hatter escorted her to the border. "You must leave Wonderland now," he said, "and return home." Alice thanked the Mad Hatter for allowing her to join the mad tea party. The Mad Hatter appeared not to hear her. "You can change your money here," he told Alice matter-of-factly as they passed the bureau-de-change at the border.

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Now in Wonderland they also use dollars and cents. One dollar there equals 100 cents, just as in the U.S. Their Wonderland dollar was originally based on the silver standard and looked similar to a U.S. silver dollar, which originally it equaled. Nowadays, it is just a paper bill that looks much like a U.S. paper dollar. However, the Wonderland dollar continued to be pegged to the silver standard, constantly increasing its value relative to the U.S. dollar. Since the price of silver fluctuated, so did the value of the Wonderland dollar relative to the U.S. dollar. To avoid these uncertainties of the market place, the Wonderland Government decided to fix the exchange rate. After consulting their national astrologers, they pegged the Wonderland dollar at exactly one Wonderland dollar to 4π U.S. dollars. Now the Wonderland cent, an unimpressive copper coin, has always been pegged at parity with the U.S. cent. Since there is no trade between Wonderland and the rest of the world, this strange arrangement did not bother the Wonderlanders. In fact, they were rather pleased with it because Wonderland tourists profit when they exchange each of their Wonderland dollars for several U.S. dollars at the border. Thus,

$$\text{1 U.S. \$} = \text{100 U.S. cents},$$
$$\text{1 WL \$} = \text{100 WL cents}$$
$$\text{1 WL cent} = \text{1 U.S. cent}$$

however,

$$\text{1 WL \$} = 4\pi \text{ US \$}$$

"But that's impossible," cried Alice. It made no sense to her. The Mad Hatter laughed and laughed and laughed his hat off. "It's really very simple," he said. "It's all in the equations. We use an irrational money system in Wonderland. By making our dollar worth more than yours, we don't have to work and can live very well too. Goodbye."

As he left, he pressed a note into Alice's hand. "Read this when you get home," he said; "then think about it till you've solved this conundrum."

When Alice arrived home from Wonderland, she read the note. It said in part: "We have discovered the secret of life without labor by multiplying the value of our dollar by a factor $\underline{S}$ relative to yours ($\underline{S}$ for silver). $\underline{S}$ originally fluctuated with the price of silver, so our Congress voted unanimously to fix the exchange rate at $\underline{S} = 4\pi$ This proved so popular that we did the same for other precious commodities simply by changing our equations. For example, your meter, kilogram, second and ampere (MKSA) system of units claims that the wave impedance of free space is $120\pi\,\Omega$, but by de-rationalizing Maxwell's equations, we made ours equal to 30 Ω. By pegging our resistance-Ω to yours as we did with the cent, while allowing our wave-impedance-Ω to change relative to yours as we did with the dollar, we made our wave-impedance-Ω worth 4π times yours." There were other examples in the Mad Hatter's note. Alice felt challenged. She thought about it long and hard till she got it. How did she solve this conundrum?

Sounds like a fairy tale? Impossible, you say? It's not. Replace Alice by Leo Young (LY) and the Mad Hatter by Professor J. A. Ratcliffe (JAR), professor of physics at Cambridge University, Cambridge, U.K., when LY was an undergraduate there attending his lectures on electromagnetism. Now JAR started his course with Coulomb's law. As in electrostatic units (esu), he did not have 4π in Coulomb's law. Instead, the 4π's pop up later in Maxwell's and other equations. Such a system of units is called "nonrationalized." (Compare the money system in Wonderland, which the Mad Hatter called "irrational.") Further, the esu system is one of several nonrationalized centimeter, gram, and second (cgs) systems (that is, systems that use cgs for their basic units, rather than meter, kilogram, and second). Thus, JAR combined a cgs system of *equations* with MKSA *units*; in other words, he taught us the nonrationalized version of the MKSA system (MKSA units with nonrationalized equations). We shall refer to the rationalized MKSA system (which is commonly used) as MKSA(r), while referring to the nonrationalized form of the MKSA system (which is almost never used, but was used in class by JAR) as MKSA(nr).

When I left Wonderland-by-the-Cam and entered the real world, working on radars and waveguides, antennas and propagation, magnetically tunable filters and microwave devices of all kinds, I was puzzled by the fact that JAR's notes made the impedance of free space equal to 30 Ω, when most engineers said it equaled $120\pi\,\Omega$, as well as by other similar discrepancies. Like Alice, I felt challenged to solve this conundrum. I treasured JAR's excellent lecture notes, but I could never tell when my answer might be off by a factor of 4π relative to almost everyone else's.

When finally I solved the puzzle, I published several short papers on it, but hardly anyone read them. I detailed my discoveries in a book (with a foreword by JAR) to favorable reviews, but hardly anyone bought it. In the 1960s and 1970s, I corresponded with and later visited the National Bureau of Standards (NBS) [now NIST (National Institute of Standards and Technology)] in Washington, DC, in particular Dr. Chester ("Chet") Page (CP). CP was very kind, even fatherly. He endorsed my work. He knew about the paradox and the mischief it could cause. He agreed that my solution cleared it up and that it would facilitate the conversion among different systems of units, not only between rationalized and nonrationalized systems, but also by extension among Gaussian, Heaviside–Lorentz, Hansen units, and the lot. However, he could not support adopting my proposal for "transitional" units. The reason he gave me was curious. He said it would make it *too easy* to convert among different systems of units! That seemed odd to me. I asked him to explain. He said that NBS's mission was to standardize. In this case, it evidently meant to standardize not only on the units, but also on the *equations*. This goal would not have been served by making it too easy to convert among different systems of units. He hoped that eventually one and only one system would prevail, namely, the rationalized MKSA system, or MKSA(r) for short. I could see his point. Nevertheless, other systems (such as a nonrationalized cgs systems) continue to thrive and do well in certain areas of physics, mostly for good reason.[1]

In the late 1960s, I worked on magnetically tunable filters. The tiny yttrium–iron–garnet (YIG) spheres were made from material specified by the supplier in a *nonrationalized* cgs system. The filter performance had to be specified in the *rationalized* MKSA system. Again, it was difficult to explain the conversion process in clear mathematical terms. Thus, just like Alice, let us see what we can do about the Mad Hatter's challenge.

Like other microwave engineers, I too use the common (rationalized) MKSA(r) system. When occasionally I need to venture into a Wonderland like "cgs-land," where people use another "currency" (unit system), I can work with them too because I know how to exchange my MKSA(r) "dollars and cents" (units) for theirs. Starting with Maxwell's equations in free space, let me give a brief explanation, at least enough to get you started.

The culprit—if there is one—is the factor 4π. If suppressed *here*, it pops up *there*. Naturally we take it out of our favorite (most often used) equations and let others worry about it elsewhere. Thus, as a microwave engineer, I will obviously take it out of Maxwell's equations before I will take it out of Coulomb's law. The unavoidable problem is one of geometry, namely, that the surface of a sphere is 4π times the square of its radius and there is nothing to be done about it. In other words, there happen to be just 4π sr in a sphere. (From here on out, we shall use the term sphere as a measure of solid angle.) Thus, if the parties have different ideas on where to place their 4π's when they meet to do business, they may use a place holder to facilitate the transition. We will call it $\underline{S}$ ($\underline{S}$ in honor of sphere or steradian or, if you are a Wonderland fan, of silver). Now, Maxwell's equations may be written (with conventional notation and symbols, but with the insertion of the place holder $\underline{S}$ that may later be set equal to either unity or 4π) as follows:

$$\nabla \cdot \mathbf{D} = \underline{S}\rho \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{H} = \underline{S}\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{4}$$

where $\underline{S} = 1$ for rationalized systems and $\underline{S} = 4\pi$ for nonrationalized systems. (Bold letters represent vectors.)

On the other hand, Coulomb's law (again with conventional symbols) becomes

$$F = \frac{\underline{S}Q_1Q_2}{4\pi\varepsilon_0\varepsilon_r r^2} \tag{5}$$

where once more $\underline{S} = 1$ gives MKSA(r) and $\underline{S} = 4\pi$ gives esu or electromagnetic units (emu) or, for that matter, MKSA(nr).

[1]Similarly, the U.S. has officially been on the metric standard since the "Mendenhall Order" of 1893 (eight years before the establishment of the NBS in 1901), but the pound, inch, etc., remain in common use—in this instance, for different reasons, part economic (by industry) and part inertia (by the public).

In rationalized systems where $\underline{S} = 1$, Maxwell's equations are free of 4π's, while in nonrationalized systems, where $\underline{S} = 4\pi$, Coulomb's law is free of 4π in the equation. Which system one picks is a matter of *convenience*—often simply the system with the fewest number of those pesty 4π's in one's equations.

The change in the numerical value of $\underline{S}$ is based on spherical geometry—on whether one implicitly uses sphere (when $\underline{S} = 1$) or steradian (when $\underline{S} = 4\pi$) as the unit of solid angle. (Since the "quantity" $\underline{S}$ is the same—only its measure changes—the unit of $\underline{S}$ is inversely proportional to its measure.) However, viewed simply as a symbol in a mathematical equation, $\underline{S}$ may conveniently be treated as a fifth base quantity (besides length, mass, time, and current) with dimensions (just like the other four quantities treated as basic).[2] Of course, its numerical value does not range over an infinite continuum of possible values (as with "real" variables like length or time); instead, it is a variable with only two possible values: unity or 4π. Inserting $\underline{S}$ needs to be only a temporary device during the conversion process (from one system of units and equations to another), a sort of place holder for either unity or 4π. Let us see how it works. Being microwave engineers, let us start with wave impedance.

Example: Treating $\underline{S}$ as a fifth base quantity, what are the dimensions of wave impedance compared to the dimensions of resistance? With the usual notation, the dimensions of resistance R are those of V/I, which we shall write $[R] = [V/I]$, using square brackets to denote dimensional equations. For wave impedance, Z_w, on the other hand, (4) yields $[Z_w] = [E/H] = [V/\underline{S}I] = [R/\underline{S}]$. Thus, the unit of Z_w is not just Ω, but "Ω per sphere" in MKSA(r) and "Ω/sr" in MKSA(nr). The conundrum is solved: "$120\pi\,\Omega$/sphere" is consistent with "30 Ω/sr."

A similar approach allows the performance of a magnetically tunable microwave filter to be expressed in MKSA(r) units when the saturation magnetization of a YIG sphere is given, as it usually is, in nonrationalized Gaussian units. (Compare Appendix B, Example 4.)

When Alice understood all this, she was very pleased with herself and wanted to tell everybody. However, no one would listen. They were too busy with their daily routines, quite happy with whatever unit system they were used to in their own laboratories [mostly MKSA(r)]. And that is the way it should be. When doing business in a strange land with a seemingly irrational currency system, we need to know the "exchange rate" for each denomination, but once back home, we will naturally revert to our own familiar currency.

A year passed. Alice missed her good friends in Wonderland and wanted to go back for a summer visit. Fortunately, she had saved a few of those precious Wonderland dollars of that irrational currency. Down she dived into the rabbit hole and quickly disappeared from view, with a new tolerance for all things that at first sight seem "irrational."

B. Vacation II

(Wherein Alice learns about symmetrical and nonsymmetrical systems of units. Symmetrical systems are systems in which ε_0 and μ_0 are both set equal to unity simultaneously, as for example, in the Gaussian system of units.)

Have you heard the latest news since Einstein? It broke recently in scientific and engineering journals, such as *Science* and *IEEE Spectrum* and it even made headlines in the popular press, as in the *San Francisco Chronicle*. Yes, light can be slowed down, even stopped! Well, it may have been news to you and me, but it wasn't news in Wonderland, where it had been known for a long time. Thus, if a Nobel prize were to be awarded for this discovery, it ought to go to a Wonderland scientist—Nobel Committee, please note. Here is the full story of how Alice first encountered this phenomenon in Wonderland. And then I will show how it explains Gaussian units, which are now in common use in Wonderland.

Last time we lost sight of Alice she had just disappeared down the rabbit hole leading to Wonderland. She had started her summer vacation and could not wait to see her old friends there again. Thus, she persuaded her parents to send her to summer camp in Wonderland. One day she was walking near camp and became lost in the nearby woods when the Cheshire Cat appeared. Alice asked the Cat for directions, but what she heard was confusing. The Cat disappeared and reappeared several times; this upset Alice, who said to the Cat (quoted in her own words by her biographer Lewis Carroll):

"'I wish you wouldn't keep appearing and vanishing so suddenly: you make one quite giddy!"

"All right," said the Cat; and this time it vanished quite slowly, beginning with the end of the tail and ending with the grin, which remained some time after the rest of it had gone.

"Well! I've often seen a cat without a grin," thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in my life!"

So how did Alice explain this curious phenomenon? Fortunately, Alice had attended classes in electrical engineering. But let me start at the beginning.

When two charges Q_1 and Q_2 exert a force on one another (like charges repel, opposite charges attract), we explain this step by step as follows. We say that the first charge emits an electric flux with a flux density symbolized by $\mathbf{D}$. This electric flux density gives rise to an electric field $\mathbf{E}$, which exerts a force $\mathbf{F}$ on the second charge that is immersed in the field $\mathbf{E}$. We may thus depict this sequence

$$Q_1 \rightarrow D \rightarrow E \rightarrow Q_2 \rightarrow F.$$

Now $\mathbf{E}$ is proportional to $\mathbf{D}$. In a vacuum, the factor of proportionality is written ε_0 so that

$$\mathbf{D} = \varepsilon_0 \mathbf{E}. \tag{6}$$

Similarly, two currents I_1 and I_2 exert forces on one another and we again explain it step by step. The first current generates a magnetic field $\mathbf{H}$. This magnetic field $\mathbf{H}$ gives rise to a magnetic flux density $\mathbf{B}$, which exerts a force on the second current that is immersed in that flux. We may thus depict this sequence

$$I_1 \rightarrow H \rightarrow B \rightarrow I_2 \rightarrow F.$$

[2]Compare Appendix A.

Now $\mathbf{B}$ is proportional to $\mathbf{H}$. In a vacuum, the factor of proportionality is μ_0 so that

$$\mathbf{B} = \mu_0 \mathbf{H}. \tag{7}$$

Equations (6) and (7) reduce the number of independent variables in Maxwell's equations by two (by connecting $\mathbf{D}$ with $\mathbf{E}$ and $\mathbf{H}$ with $\mathbf{B}$). Equations (6) and (7) are referred to as the "constitutive relations."

It can be shown from Maxwell's equations and from the constitutive relations that there exist electromagnetic waves that travel with a velocity c in vacuum,[3] given by

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \tag{8}$$

The values of ε_0 and μ_0 can be measured experimentally (or defined *ab initio*) and the velocity of the electromagnetic waves can thus be calculated from (8). (One of the early triumphs of Maxwell's equations was that the velocity of light was the same as the velocity predicted by (8), the first credible evidence that light was an electromagnetic wave.)

It is, of course, a nuisance to have to remember the numerical values of ε_0 and μ_0. Their numerical values are a function of the basic units chosen (e.g., centimeter and second) and of the constants we introduce into Maxwell's equations. In MKSA rationalized units, all the constants in Maxwell's equations are set equal to unity, thus simplifying the equations, but complicating the numerical values of ε_0 and μ_0. Why not start from a different point-of-view, i.e., start by setting often used constants like ε_0 and μ_0 equal to unity and let the chips (i.e., nonunity constants) fall where they might in the equations?

That is precisely what was done by the innovative Wonderland Institute of Standards and Technology (known fondly as WIST), which functions much like NIST in the U.S. WIST made a bold move to give Wonderland scientists and engineers a leg up, so to speak, over those in the rest of the world. On a trial basis, WIST started with the constants in (6) and (7) by setting them equal to unity, thus making $\varepsilon_0 = 1$ and $\mu_0 = 1$. This reduced the arithmetic workload on Wonderland scientists and gave them more time for thinking. This system of units was called the WIST system.

Now WIST also standardized on cgs (rather than on MKSA) units. By not tampering with the constants in Maxwell's equations, WIST retained (8) unchanged. A curious by-product of this WIST system was that by (8), the velocity of light became unity, thus slowing the velocity of light in Wonderland to just 1 cm/s. This, in turn, had some interesting consequences for which Wonderland became famous. One of them was the grin of the Cheshire Cat as related in the quotation cited above. It works as follows.

The Cat, being a shy animal, was always ready to vanish by bolting at the slightest provocation (real or imagined) from anyone. The Cat would always crouch ready for a quick getaway, tip of tail nearest the stranger, head farthest away, ready to run, but always keeping the stranger in view. In this way, the Cat observed Alice's every move, grinning, but eyeing her suspiciously.

This Cat was just 60-cm long from head to tail; it now took light exactly 1 min to travel the length of the beast. Thus, when the animal jumped up suddenly and took flight, the light originating from the tail, having arrived at the viewer's eyes first, made the tail disappear first. The light from the grinning face, having taken 1 min longer to arrive, kept coming for another minute after the tail had disappeared and so the grin stayed 1 min longer and for a while appeared to be without body. Thus, you had a phenomenon for which Wonderland became famous. It was a strange sight to behold and a great tourist attraction, that disembodied grin of the Cheshire Cat.

However, there was a downside too. It made it difficult for Wonderland scientists and engineers to work with their colleagues from other countries where the velocity of light (c) was so much greater (approximately 3×10^{10} cm/s in vacuum).

Thus, WIST ordered Maxwell's equations to be outfitted with a "place holder," the constant $\underline{U}$, equal to unity in Wonderland, but outside Wonderland equal to $1/\underline{c}$, where $\underline{c}$ is the numerical value of c measured in centimeters per second. Retaining the other constant $\underline{S}$ already explained and adding $\underline{U}$, Maxwell's equations (1)–(4) now become

$$\nabla \cdot \mathbf{D} = \underline{S}\rho \tag{9}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{10}$$

$$\underline{U}^{-1} \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{11}$$

$$\underline{U}^{-1} \nabla \times \mathbf{H} = \underline{S}\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{12}$$

where $\underline{S} = 1$ for rationalized (r) systems and $\underline{S} = 4\pi$ for nonrationalized (nr) systems. Further, $\underline{U} = 1$ for both MKSA (rationalized and nonrationalized) systems and in the WIST system. However, in the rest of the world, $\underline{U} = 1/\underline{c}$, whenever $\varepsilon_0 = 1$ and $\mu_0 = 1$, where it is known as the Gaussian system of units ($\underline{S} = 4\pi$, $\varepsilon_0 = 1$, $\mu_0 = 1$, $\underline{U} = 1/\underline{c}$, plus cgs base units) after the famous 17th–18th Century German mathematician and physicist Johann Carl Friedrich Gauss. Thus was the Gaussian system independently invented by the ingenious people at WIST. With the "place holder" constant $\underline{U}$ in Maxwell's equations, the velocity of light is derived to be

$$c = \frac{1}{\underline{U}\sqrt{\varepsilon_0 \mu_0}}. \tag{13}$$

Thus, while inside Wonderland, WIST set all three constants ε_0, μ_0, and $\underline{U}$, equal to unity, yet when working with scientists outside Wonderland, WIST allowed the use of the Gaussian system in which $\underline{U} = 1/\underline{c}$, so as to retain both $\varepsilon_0 = 1$ and $\mu_0 = 1$ along with the real-world velocity of light.

Summing up, three systems of electromagnetic units are of special interest. They are defined by the numerical values of four constants in Maxwell's equations and in the constitutive relations (1)–(6), as shown in Table I.

Clearly the WIST system appears superior, having all but one of these constants equal to unity. The Wonderland Congress was lobbied to pass a law making the experiment official. Thus, the

[3] c stands for the velocity of light, a dimensional (length/time) number equal to 199 792 458 mm/s, or approximately 3×10^{10} cm/s. We shall have many occasions where we shall need a pure (dimensionless) number equal to the velocity of light in centimeters per second; we shall denote the latter by $\underline{c}$ (to distinguish it from the former, denoted by c).

TABLE I
CONSTANTS OF THE MKSA(r), GAUSSIAN, AND WIST SYSTEMS COMPARED

	$\underline{S}$	$\underline{U}$	ε_0	μ_0
MKSA(r)	1	1	$\sim 10^{-9}/36\pi$	$4\pi\times 10^{-7}$
Gaussian	4π	$\sim 1/3\times 10^{10}$	1	1
WIST	4π	1	1	1

Wonderlanders discovered that the velocity of light could be reduced to a mere dribble of 1 cm/s. However, there was a catch.

Although scientists in Wonderland were happy with the WIST system of units, traffic engineers were not at all happy with the accidents that ensued! Cars driving at speeds greater than 1 cm/s constantly bumped into one another before they could see each other coming. So another law had to be passed that no one was to move faster than 1 cm/s. Wonderland almost came to a standstill. The laws were repealed. The experiment ended. The Gaussian system it was that survived!

By what names shall we know the units of $\underline{U}$? Physically, $\underline{U}$ arises from planar angle (from turns or windings), as does $\underline{S}$ from solid angle (from flux). We shall name the unit of $\underline{U}$ in the MKSA systems the *turn* (compare *sphere* for $\underline{S}$); in Gaussian units, we will name the unit of $\underline{U}$ the *curl* because a *curl* (think of a *coil*, if you please) is many *turns*. The *curl* is larger than the *turn* (by a factor $\underline{c}$) because the numerical value of $\underline{U}$ is smaller in Gaussian units than it is in MKSA units. (A *larger* measure of a *constant* physical quantity implies that its unit is *smaller* and vice versa.) We will see in Appendix B why it helps to give names (in this case, *turn* and *curl*) to all units (as we give names to all people).

Alice enjoyed her adventures in summer camp. Where but in Wonderland could she have experienced such a startling experiment with the velocity of light? She was excited to have gained a clearer understanding of units and equations in electrical engineering. As she left Wonderland for home, she imagined the Cheshire Cat's famous grin grinning her a farewell from the tree tops, reminding her of the interesting lesson she had just learned. She could hardly wait to get back to school and tell her teacher all about Gaussian units.

C. Vacation III

(Alice bears witness in a royal Court of Law and shows that a diversity of unit systems is no threat to anyone.)

The trial was a scandal. Alice was called as an expert witness. The King of Hearts was the judge. The Queen of Hearts was the prosecutor. The Knave of Hearts was in chains, guarded by two aces with long lances, accused of being a traitor for trying to introduce rationalized units into the nonrationalized Kingdom of Hearts. Alice was called to testify for the Defense. She towered over the card characters and so was not afraid to speak up. "Where is the King of Spades? Where are the other kings?" she asked. "Why haven't they been called as witnesses?" Then she added pointedly, "And where are the Jokers?"

You see, Wonderland was divided into four kingdoms just like a pack of cards is divided into four suits. The kingdoms were ruled, respectively, by the King of Spades, the King of Hearts, the King of Diamonds, and the King of Clubs. The King of Spades was the most senior, followed in order of seniority by the Kings of Hearts, Diamonds, and Clubs. These four kingdoms cooperated, argued, squabbled, made alliances and fought among themselves, even trumped one another. It was very confusing to Alice.

TABLE II
LINE-UP OF THE CARDS

$\underline{S}$ \ $\underline{U}$	$\underline{S} = 1$ [rationalized system]	$\underline{S} = 4\pi$ [non-rationalized system]
$\underline{U} = 1$ [nonsymmetrical system]	♠ **Spades** [MKSA(r)]	♥ **Hearts** [MKSA(nr)]
$\underline{U} = 1/\underline{c}$ [symmetrical system]	♣ **Clubs** [Heaviside-Lorentz]	♦ **Diamonds** [Gaussian]

In each kingdom there were two cards like no other. They were the Wonderland equivalent of diplomats. These "Jokers" could act as brokers or peace makers. They could represent any other card in their kingdom, thereby settling disagreements, arranging alliances, making connections, and generally smoothing things out. Since there were only two of them in each kingdom and the kings were prone to squabble, they were kept very busy. Thus, they were constantly in demand, moving from one place to another, leaving the scene as soon as they had settled a dispute.

One of the causes of friction or miscommunications among these four kingdoms was that each king—for reasons of historical continuity, or out of a sense of false pride, or from mere pigheadedness, or simply through inertia—favored a different system of units and equations. We already saw in (9)–(12) that when two numerical constants in Maxwell's equations are replaced by algebraic symbols, the equations can represent more than one system of units. Each of the four kingdoms of cards espoused one system of units and only one, as summarized in Table II.

Thus, Spades favored the rationalized form of the MKSA system, or MKSA(r) for short, while Hearts preferred the nonrationalized MKSA system, or MKSA(nr) for short. Diamonds and Clubs had strong lobbies of theoretical physicists who frequently used the electric constant ε_0 and the magnetic constant μ_0 in their equations and could save ink by setting $\varepsilon_0 = \mu_0 = 1$; therefore, they lobbied hard for Gaussian and Heaviside–Lorentz[4] units, respectively; Diamonds went Gaussian, while Clubs chose Heaviside–Lorentz.

Thus, the two heart-shaped symbols (Spades and Hearts) agreed on MKSA and $\underline{U} = 1$, but not on rationalization, while the other two (Diamonds and Clubs) agreed on cgs and $\varepsilon_0 = \mu_0 = 1$, but again could not agree on rationalization. On the other hand, the two "black" kingdoms agreed to rationalize, while the two "red" kingdoms agreed not to rationalize. These royal idiosyncrasies are summarized in Table II.

[4]The Heaviside–Lorentz system of units is the rationalized version of the Gaussian system, i.e., the only change is $\underline{S} = 1$ instead of $\underline{S} = 4\pi$. Both systems assign the numerical value unity to both the electric and magnetic constants ($\varepsilon_0 = 1$ and $\mu_0 = 1$) and are, therefore, called "symmetrical." Systems in which one or both of these two constants are not equal to unity are called nonsymmetrical.

Such disagreements made scientific and engineering cooperation among the four kingdoms difficult and became the source of much friction. Whenever there was a big problem, the Jokers were summoned to make peace. No worry, no war. However, what is that all got to do with units?

Well, the Jokers stand for "transitional units" and "transitional equations," i.e., units and equations with augmented dimensions in $\underline{S}$ or $\underline{U}$. To convert from one system of units or equations to another, follow this three-step process.

Step 1) Starting in the first (originating) system, create a new quantity or equation that has been augmented by dimensions in $\underline{S}$ and $\underline{U}$ so as to be dimensionally complete. This is equivalent to substituting the Joker for the original card, whatever the suit, since the transitional form, like the Joker, can speak the language of any system (kingdom).

Step 2) Make sure that this quantity or equation (now expressed in "transitional" form) is still correct numerically in the original system when $\underline{S}$ and/or $\underline{U}$ are replaced by their original values.

Step 3) Replace $\underline{S}$ and $\underline{U}$ by their numerical values in the final "target" system.

Examples are worked out in detail in Appendixes A and B.

Alice pointed out to the court that the system used by the King of Spades was no threat to the King of Hearts and demanded that Spades be called to testify as to their peaceful intentions. She particularly urged the court to bring in the Jokers so they could explain how they enable citizens from different kingdoms to communicate harmoniously.

The Queen of Hearts was outraged. "Off with her head," she screamed at Alice, who was not intimidated. Alice knew the Queen always forgot her threat as soon as it was uttered and nobody took her seriously anyway. It remained to convince the King of Hearts who presided over the court that his throne was in no danger. Alice pointed out that it was certainly advantageous for everyone in the kingdom to use the same system of units and equations, but also that other systems in use in other kingdoms were no threat to his royal Highness. She even suggested that the number of Jokers in each pack of cards be increased to more than two! The King pointed out that it was state policy for every king never to agree to anything proposed by another king, which made it extremely unlikely that they would all agree on Alice's suggestion. Alice requested that the Knave of Hearts be set free, to which the King agreed.

Alice moved a vote of thanks to the Jokers. The motion was passed unanimously and the court was dismissed.

The cards went home to play.

Appendix A
How to Convert Equations From One Unit System to Another

Example #1

Given that the force per unit length dF/dL between two infinitely long thin straight conductors carrying currents I_1 and I_2, spaced a distance r apart, is in MKSA(r) units

$$\frac{dF}{dL} = \frac{\mu_0 I_1 I_2}{2\pi r} \tag{A1}$$

what is it in Gaussian units?

We must express every quantity in the equation dimensionally including the two dimensions of $\underline{S}$ and $\underline{U}$. (The four dimensions of length, mass, time, and current are not necessary since they do not affect the equations.) As is evident from (A1), the only quantity we need to worry about is μ_0. Now

$$B = \mu_0 H \tag{A2}$$

so the dimensions of μ_0 are those of B/H, which we write

$$[\mu_0] = \frac{[B]}{[H]} \tag{A3}$$

where the square brackets denote that (A3) is a dimensional equation.

Since we are concerned only with the dimensions of μ_0 in terms of $\underline{S}$ and $\underline{U}$ (and do not need to supply the dimensions of length, mass, time, and current), it follows from the two Maxwell's equations (11) and (12) that

$$[\mu_0] \propto \left[\underline{S}^{-1}\right] \left[\underline{U}^{-2}\right]. \tag{A4}$$

Further, since in the MKSA(r) system of units $\underline{S} = 1$ and $\underline{U} = 1$, (A1) is balanced dimensionally by multiplying the right-hand side of (A1) by $\underline{SU}^2$. Thus,

$$\frac{dF}{dL} = \frac{\underline{SU}^2 \mu_0 I_1 I_2}{2\pi r} \tag{A5}$$

which does not change the numerical balance of the MKSA(r) equation because $\underline{S} = 1$ and $\underline{U} = 1$; but when the Gaussian values of $\underline{S} = 4\pi$ and $\underline{U} = 1/\underline{c}$ are substituted, the equation becomes

$$\frac{dF}{dL} = \frac{2\mu_0 I_1 I_2}{\underline{c}^2 r} \tag{A6}$$

where $\underline{c} = 3 \times 10^{10}$ (approximately), the velocity of light in vacuum expressed in centimeters per second.

Example #2

Given (A6) written in Gaussian units, what is the correct equation in MKSA(r) units?

Take the steps just outlined in reverse. First, find the $\underline{S}$ and $\underline{U}$ dimensions of μ_0 as before, which brings us to (A4) again. Derive (A5), remembering that substituting the numerical values $\underline{S} = 4\pi$ and $\underline{U} = 1/\underline{c}$ in Gaussian units, (A5) must reduce to the given (A6). Finally, replace $\underline{S}$ and $\underline{U}$ with their numerical values in MKSA(r) units, namely, $\underline{S} = 1$ and $\underline{U} = 1$ and obtain (A1).

Appendix B
How to Compare Units in One System to Units in Another

Example #3

Compare the units of current in the MKSA(r) and the Gaussian systems.

The defining equation for current in a standards laboratory is based on (A5), i.e., the current standard is determined by measuring the force between two identical currents in a current balance. Thus, we shall start with that equation.

Setting $I_1 = I_2 = I$, (A5) becomes

$$\frac{dF}{dL} = \frac{\underline{S}U^2\mu_0 I^2}{2\pi r}. \tag{B1}$$

In the MKSA(r) system, μ_0 is *defined* to be equal to $4\pi \times 10^{-7}$. Now we have to compare the units of μ_0 in MKSA(r) and Gaussian units. Since a physical quantity (such as μ_0 or any other) does not change with the unit system—only its *measure* changes—the larger the measure, the smaller the unit and vice versa. In Gaussian units, μ_0 equals unity, therefore, the ratio of an MKSA(r) unit of μ_0 to a Gaussian unit of μ_0 equals $10^7/4\pi$. The ratio of the units of $\underline{S}$, sphere/steradian, is 4π. The ratio of the units of $\underline{U}$ is 3×10^{10} approximately. Length cancels. Finally, for force the ratio is newton/dyne, which equals 10^5. Now let us write (B1) twice, once in MKSA units and once in Gaussian units, and then divide the respective sides of the two equations (length cancels) and, thus, find the ratio of the two units of current as follows:

One ampere/one Gaussian unit of current (Franklin/second)

$$\begin{aligned} &= \frac{\{\text{newton/dyne}\}^{1/2}}{\left\{(\text{sphere/steradian})\times(\text{turn/curl})^2\times\left(\dfrac{\langle\mu_0(M:r)\rangle}{\langle\mu_0(g)\rangle}\right)\right\}^{1/2}} \\ &\approx \frac{10^5}{\left[4\pi\times\left(\dfrac{1}{3}\times 10^{10}\right)^2\times\left(\dfrac{10^7}{4\pi}\right)\right]^{1/2}} \\ &= \left\{\frac{9\times 10^{25}}{10^7}\right\}^{1/2} \\ &= 3\times 10^9 \end{aligned} \tag{B2}$$

where $\langle\mu_0(M:r)\rangle$ and $\langle\mu_0(g)\rangle$ stand for the units of μ_0 in MKSA(r) and Gaussian units, respectively. The ratio was determined in (B2) by remembering that the *size* of the unit of a physical constant is inversely proportional to the *numerical value* of that constant; since $\mu_0(M:r) = 4\pi/10^7$ and $\mu_0(g) = 1$, therefore, the ratio $(\langle\mu_0(M:r)\rangle/\langle\mu_0(g)\rangle)$ plugged into (B2) is $10^7/4\pi$. Similarly, the curl is larger than the turn by a factor of $\underline{c}$, namely, $\sim 3 \times 10^{10}$. Thus, 1 amp equals $\sim 3\times 10^9$ Gaussian units of current (3×10^9 Franklins/s).

Example #4

You want to design a magnetically tunable filter using a YIG sphere. The supplier of the sphere specifies the saturation magnetization M_S of the material in gauss—and with a 4π multiplier added—thus: $4\pi M_S = 1750$ G. However, you need to specify filter performance in MKSA(r) units. You want to convert magnetic field H and magnetization M from Gaussian to MKSA(r) units.

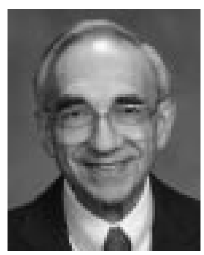
To Convert Units of Magnetic Field: From (12), $[H\} = [\underline{SU}I/L]$. Since 1 sphere $= 4\pi$ sr, 1 curl $= \underline{c}$ turns, then from (B2), the units of H are related by

$$1\ \text{amp/m} = 4\pi \times 10^{-3}\ \text{Oe}. \tag{B3}$$

To Convert Units of Magnetization: Magnetization is defined by the International Electrotechnical Commission (IEC) and the IEEE by $B = \mu_0(H + \underline{S}M)$,[5] which gives M dimensions of $[M] = [\underline{S}^{-1}H]$. Hence (before the addition of angle dimensions), M is measured in amperes per meter in MKSA(r) and in oersted in Gaussian units. However, in physics, M is usually defined by another equation, i.e., $B = \mu_0 H + \underline{S}M$,[6] which would make $[M] = [\underline{S}^{-1}B]$. Thus, electrical engineers treat magnetization M dimensionally proportional to magnetic field H, while most physicists do so with respect to magnetic induction B. To conform to the IEC/IEEE definition, one substitutes oersted for gauss before converting M from Gaussian to MKSA(r) units. (This is possible because $\mu_0 = 1$ in Gaussian units.) The conversion ratios for M and for H now differ only by the ratio steradian/sphere. Hence, from (B3), the units of M are

$$1\ \text{amp}/(\text{m}\cdot\text{sphere}) = 10^{-3}\ \text{Oe/sr}. \tag{B4}$$

Thus, without showing angle dimensions, (B3) and (B4) would be inconsistent.



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[5] We have added the $\underline{S}$ to include both rationalized and nonrationalized units.

[6] We have added the $\underline{S}$ to include both rationalized and nonrationalized units.